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On the quark distribution in an on-shell heavy quark and its all-order relations with the perturbative fragmentation function

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Abstract: I present new results on the quark distribution in an on-shell heavy quark in perturbative QCD and explore its all-order relations with heavy-quark fragmentation. I first compute the momentum distribution function to all orders in the large-β_0 limit and show that it is identical to the perturbative heavy-quark fragmentation function in the same approximation. I then analyze the Sudakov limit of the distribution and the fragmentation functions using Wilson lines and prove that the corresponding Sudakov exponents in the non-abelian theory are the same to any logarithmic accuracy. The anomalous dimension is then determined to two-loop order, corresponding to next-to-next-to-leading logarithmic accuracy in the exponent, in two ways: the first by extracting the singular terms from a recent calculation of the fragmentation function and the second by performing the two-loop Wilson-line calculation in configuration space. I find perfect agreement between the two.

Keywords: Renormalization Regularization and Renormalons, Heavy Quarks Physics, QCD, NLO Computations.

Contents

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1. Introduction

Relations between space-like processes that probe the parton distribution function and time-like processes involving fragmentation have been known since long. This includes the Drell-Yan-Levy relation between deep-inelastic structure functions and single-particle inclusive cross section in e^+e^- annihilation [1], as well as the Gribov-Lipatov reciprocity relation [2], relating the space-like and time-like splitting functions that determine the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [3] evolution of parton densities and fragmentation functions, respectively. In dimensional regularization, the latter relation is violated beyond the leading order.

In this paper we find and explore new all-order perturbative relations between distribution and fragmentation functions, which are specific to heavy quarks.[1] When dealing with heavy quarks one can define and compute perturbative distribution and fragmentation functions, replacing the hadronic states by ones composed of on-shell quarks and gluons. Owing to collinear singularities this is not possible for light quarks, where only the ultraviolet (DGLAP) evolution of these functions can be computed in perturbation theory, while the actual matrix elements, which set the initial condition for the evolution, can only be defined non-perturbatively. For heavy quarks the mass effectively cuts off collinear radiation and, if $m \gg \Lambda$, it provides a hard scale for the coupling. This makes the perturbative initial condition for DGLAP evolution of the distribution and fragmentation functions well defined to any order in perturbation theory. These perturbative initial conditions are the subject of this paper. Of course, these functions still differ from their non-perturbative counterparts by power corrections, which we shall briefly discuss as well.

The non-perturbative process-independent definitions of distribution and fragmentation functions [4, 5] — see eqs. (2.1) and (3.30), respectively — apply to both light and heavy quarks. These field-theoretic definitions are based on the Fourier transform of certain non-local matrix elements on the lightcone. The distribution function $f(x;\mu)$ measures the longitudinal momentum distribution of a quark in a hadron, where the quark momentum fraction is $x \in [0,1]$. This function can be interpreted as a probability distribution, barring the fact that it requires renormalization. Similarly, the fragmentation function $d(x;\mu)$ measures the probability distribution of producing a hadron with fraction x of the momentum originally carried by the quark. Also here $x \in [0,1]$. Using the heavy quark expansion [6] one can show that for $\mu \sim m$ both $f(x;\mu)$ and $d(x;\mu)$ peak at $1-x = \mathcal{O}(\Lambda/m)$.

The momentum distribution function of a heavy quark in a heavy meson plays a central role in the calculation of inclusive decay spectra such as $\bar{B} \longrightarrow X_s\gamma$ and semileptonic B decays [7]–[14]. This distribution essentially determines the shape of inclusive decay spectra in the experimentally-important endpoint region. In this kinematic domain the invariant mass of the produced hadronic system is small compared to the quark mass and the $\mathcal{O}(\Lambda/M)$ fraction of the momentum which is carried by the light degrees of freedom in the meson becomes important. In this context it is convenient to analyze the quark distribution function in the infinite-mass limit, where it is often called the "Shape Function".

Although the quark distribution in a heavy meson is a non-perturbative distribution, it has an important perturbative ingredient. Being completely inclusive, this distribution can be approximated by the quark distribution in an on-shell heavy quark $f_{\rm PT}(x;\mu)$, while non-perturbative effects are treated as corrections. This idea is theoretically appealing, however, it is not easy to achieve. First, the perturbative description of the quark distribution in an on-shell heavy quark can only be reliable if Sudakov logarithms are resummed, resummation that takes the form of exponentiation in moment space. Moreover, running-coupling effects are significant and lead to infrared renormalons in these moments. The corresponding ambiguities are only resolved at the non-perturbative level, making the

[1] We do not consider in this paper deep inelastic structure functions.

resummation of running-coupling effects necessary for systematic separation between perturbative and non-perturbative contributions. The Dressed Gluon Exponentiation (DGE) approach [14, 15, 16, 30, 17] incorporates renormalon resummation in the calculation of the Sudakov exponent. This opens up the way for consistently using the resummed quark distribution in an on-shell quark as the baseline for describing the quark distribution in a heavy meson.

In a completely different application of QCD, namely heavy-quark production in hard scattering processes, one encounters the heavy-quark fragmentation function, $d(x;\mu)$. Also here the resummed perturbative initial condition for the evolution, $d_{\rm PT}(x;\mu)$, provides a baseline for a systematic description of the non-perturbative object [27]–[31]. DGE-based predictions [30], which involve just one or two non-perturbative power corrections, have proven successful in describing the inclusive b production cross section measured at LEP.

In this paper we compute the quark distribution in an on-shell heavy quark, $f_{\rm PT}(x;\mu)$, in two different limits. In each case a certain gauge-invariant set of radiative corrections is controlled to all orders. The two limits are:

- The single-dressed-gluon approximation (section 2). Taking the large-β_0 limit — or, formally, the large-N_f limit — we perform an all-order resummation of running coupling effects (renormalons). This resummation has two applications [19, 18, 20]: first, it improves the leading-order calculation incorporating BLM-type radiative corrections to all orders [21, 22], and second, it probes the structure of power corrections.

- The $x \longrightarrow 1$ Sudakov limit (section 3). Taking the large-x limit we keep only singular terms in $f_{\rm PT}(x;\mu)$, which build up the Sudakov exponent. Here the calculation is done to two-loop order, corresponding to next-to-next-to-leading logarithmic (NNLL) accuracy in the exponent. More generally, following the work of Korchemsky and Marchesini in ref. [23], we show that the singular $x \longrightarrow 1$ terms are fully captured by the distribution defined in the $m \longrightarrow \infty$ limit, the so-called "Shape Function". This means that the exponent is computable, to all orders, through the renormalization of a Π-shaped Wilson-line operator[2] with two antiparallel timelike rays, connected by a lightlike segment.

Interestingly, in both limits we find that exactly the same resummed expressions hold for the perturbative heavy-quark fragmentation function [29, 30, 31]. In general though, the two functions do differ; for example, their DGLAP evolution away from the large-x limit starts differing at two loops [2].

[2]Recall that not all Sudakov anomalous dimensions can be computed in the Eikonal approximation. For example, the jet function controlling the large-x limit of deep-inelastic structure functions [24, 25, 16, 26], which is sensitive to collinear radiation from a light quark, cannot be reproduced in this approximation.

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2. Quark distribution in an on-shell heavy quark computed with a single dressed gluon

2.1 Definition and calculation of the gluon emission cut

We define the parton distribution [4] in a heavy meson $|H(p)\rangle$ in the standard way, by

$$f(x;\mu)=\int_{-\infty}^{\infty}\frac{dy^-}{4\pi}\,\mathrm{e}^{-ixp^+y^-}\;\langle H(p)|\,\bar{\Psi}(y)\Phi_y(0,y)\gamma_+\Psi(0)\,|H(p)\rangle_\mu\,,\qquad(2.1)$$

where μ is introduced as an ultraviolet renormalization scale for the operator, y is a light-like vector in the "$-$" direction, $\Phi_y(0,y)$ is a path-ordered exponential in this direction connecting the points y and 0, i.e.

$$\Phi_y(0,y)\equiv\mathbf{P}\exp\left(ig\int_0^y dz_\mu A_\mu(z)\right).$$

Some properties of $\Phi_y(0,y)$ are collected in appendix A.

In order to compute the matrix element it is convenient use the lightcone gauge, where

$$\sum_\lambda \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)*}=-g_{\mu\nu}+\frac{k_\mu y_\nu+k_\nu y_\mu}{k\cdot y}\,.\qquad(2.2)$$

Then, at one loop there is only one diagram: the box. The calculation, based on of eq. (2.1) is summarized in appendix A. In $4-2\epsilon$ dimensions the result for the not-yet-renormalized quark distribution is:

$$f_{\rm PT}^{(4-2\epsilon)}(x;\mu)=\delta(1-x)\left[1+\mathcal{O}(\alpha_s)\right]+\left(\frac{\mu^2\mathrm{e}^{\gamma_E}}{m^2}\right)^\epsilon\frac{C_F}{\beta_0}\int_0^\infty du\,T(u)\,\left(\frac{\Lambda^2}{m^2}\right)^u\,B(x,u,\epsilon)\quad(2.3)$$

with

$$B(x,u,\epsilon)=\frac{\Gamma(\epsilon+u)}{\Gamma(1+u)}\,\mathrm{e}^{\frac{5}{3}u}\,(1-x)^{-2\epsilon-2u}x^u\,\left[(1-u-\epsilon)\frac{x}{1-x}+\frac{1+u}{2}(1-x)\right],\qquad(2.4)$$

where we used the Borel-modified gluon propagator $1/(-k^2)\longrightarrow 1/(-k^2)^{1+u}$ in order to resum running-coupling effects of all orders. With one-loop running coupling $T(u)=1$. To go beyond one-loop running we use the scheme-invariant Borel representation [32] where $T(u)$ is the Laplace transform of the 't Hooft coupling:

$$\begin{aligned}A(\mu)&=\frac{\beta_0\alpha_s^{\text{'t Hooft}}(\mu)}{\pi}=\int_0^\infty du\,T(u)\,\left(\frac{\Lambda^2}{\mu^2}\right)^u;\qquad \frac{dA}{d\ln\mu^2}=-A^2(1+\delta A)\,,\\ T(u)&=\frac{(u\delta)^{u\delta}\mathrm{e}^{-u\delta}}{\Gamma(1+u\delta)};\qquad \ln\left(\frac{\mu^2}{\Lambda^2}\right)=\frac{1}{A}-\delta\ln\left(1+\frac{1}{\delta A}\right)\end{aligned}\qquad(2.5)$$

with $\delta\equiv\beta_1/\beta_0^2$, where $\beta_0=\frac{11}{12}C_A-\frac{1}{6}N_f$ and $\beta_1=\frac{17}{24}C_A^2-\frac{1}{8}C_FN_f-\frac{5}{24}C_AN_f$. We define Λ in $\overline{\rm MS}$.

Out of the terms in the square brackets in eq. (2.4) only $\frac{x}{1-x}$ (with a coefficient 1) comes from the Axial gauge part of the propagator in eq. (2.2), which is proportional to $1/k\cdot y$. Through the y^- and the momentum integration this singularity of the propagator is converted into a $1/(1-x)$ singularity, $k\cdot y/p\cdot y\longrightarrow(1-x)$. The square brackets are a generalization of the splitting function. It is noted that eq. (2.3) is *identical* to the *perturbative fragmentation function* [29] computed in the same approximation in ref. [30] using a different technique.

2.2 Moments and evolution

In eq. (2.3) we computed only the real gluon emission contribution. Clearly this result is singular at $x \longrightarrow 1$. Virtual corrections generate $\mathcal{O}(\alpha_s)$ divergent contributions, which are proportional to $\delta(1-x)$. Owing to the inclusive nature of eq. (2.1) infrared singularities cancel out. However, there are also ultraviolet singularities. We therefore proceed by first taking a logarithmic derivative with respect to the mass to remove the ultraviolet divergence. Then we can go to four dimensions setting $\epsilon = 0$. Next, we go to moment space,

$$F_N = \int_0^1 dx x^{N-1} f(x;\mu)\,, \tag{2.6}$$

where we can use the fact that the first moment $F_{N=1}$ corresponds to the total number of quarks in the quark, a conserved current, so it is $F_{N=1} = 1$ for any mass and the derivative must vanish identically. This allows us to reconstruct the virtual (N-independent) terms missing in eq. (2.3) out of the real-emission (N-dependent) ones. We get:

$$\frac{d\ln F_N}{d\ln m^2} = -\frac{C_F}{\beta_0}\int_0^\infty du\; T(u)\,\left(\frac{\Lambda^2}{m^2}\right)^u\, B_F(N,u)\,, \tag{2.7}$$

with

$$\begin{aligned} B_F(N,u) &= \int_0^1 dx\,\left(x^{N-1}-1\right)\, B(x,u,\epsilon=0)\\ &= \mathrm{e}^{\frac{5}{3}u}\Bigg[(1-u)\Gamma(-2u)\left(\frac{\Gamma(N+1+u)}{\Gamma(N+1-u)} - \frac{\Gamma(2+u)}{\Gamma(2-u)}\right) + \\ &\quad + \frac{1}{2}(1+u)\Gamma(2-2u)\left(\frac{\Gamma(N+u)}{\Gamma(N+2-u)} - \frac{\Gamma(1+u)}{\Gamma(3-u)}\right)\Bigg] + \mathcal{O}\left(\frac{u}{\beta_0}\right). \end{aligned} \tag{2.8}$$

Next, we would like to integrate over $\ln m^2$ to recover F_N. However, this brings back the ultraviolet divergence which takes the form of a $1/u$ singularity. Therefore, we must perform ultraviolet subtraction. The result is:

$$F_N^{\rm PT}(m;\mu_F) = 1 + \frac{C_F}{\beta_0}\int_0^\infty \frac{du}{u}\,T(u)\left(\frac{\Lambda^2}{m^2}\right)^u\left[B_F(N,u) + \left(\frac{m^2}{\mu_F^2}\right)^u B_{\rm AP}(N,u)\right] + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \tag{2.9}$$

where μ_F is a factorization scale and $B_{\rm AP}(N,u)$ is the (non-singlet) Altarelli-Parisi evolution kernel,

$$B_{\rm AP}(N,u) = \sum_{n=0}^\infty \frac{\gamma_n(N)\, u^n}{n!}\,. \tag{2.10}$$

The leading order coefficient in (2.10) is renormalization-scheme invariant, and it equals to the $u=0$ limit of (2.8), ensuring the cancellation of the $1/u$ singularity in (2.9):

$$\gamma_0(N) = S_1(N) - \frac{3}{4} + \frac{1}{2}\left(\frac{1}{N+1} - \frac{1}{N}\right),$$

where $S_k(N) \equiv \sum_{j=1}^N 1/j^k$, so $S_1(N) = \Psi(N+1) + \gamma_E$. In the $\overline{\rm MS}$ factorization scheme $\gamma_n(N)$ are known to NNLO ($n=2$) in full [33] and to all orders in the large-β_0 limit [34], see appendix C.

We find that the final result for the moments of the quark distribution in the large-β_0 limit, given by eq. (2.9), is identical to the one corresponding to the heavy-quark fragmentation in the same approximation, see eqs. (38) and (39) in ref. [30]. The expansion of $F_N^{\rm PT}(m;\mu_F)$ in powers of α_s can be readily obtained from eq. (2.9) by expanding the terms in the square brackets in powers of u and integrating order by order. The result, to $\mathcal{O}(\alpha_s^2)$, appears in eq. (43) of ref. [30].

The infrared-renormalon structure can be read off eq. (2.8): there are simple poles at all integer and half integer values of u, except for $u = 1$. Note that all these singularities are absent in the real-emission expression in x space, where one finds just an upper limit of the form $(1-x) \gtrsim \Lambda/m$ on the values of x for which the Borel integral exists. These renormalons are all associated with the integration over x for $x \longrightarrow 1$, and within the scheme-invariant formulation of the Borel transform (where β_1-terms are factored out) they are expected to remain simple poles in the full theory. As already noted in refs. [30, 14], the residues in eq. (2.8) depend on N such that at large N ambiguities appear as integer powers of $(N\Lambda/m)$. These parametrically-enhanced power terms are related to renormalon ambiguities in the Sudakov exponent, and they will be discussed further in the next section.

3. Sudakov logs in the quark distribution function

3.1 Definitions and results

In the large-N limit Sudakov logarithms exponentiate as follows:

$$F_N^{\rm PT}(m;\mu_F) = \tilde{H}(m;\mu_F)\,\tilde{S}_N(m;\mu_F) + \mathcal{O}\left(\frac{1}{N}\right), \tag{3.1}$$

where

$$\begin{aligned}\tilde{S}_N(m;\mu_F) &= \exp\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_s(m)}{\pi}\right)^n\left[\sum_{k=1}^{n+1} C_{n,k}\ln^k N + \mathcal{O}(1)\right]\right\} \\ &= \exp\left\{\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\left[\int_{(1-x)m}^{\mu_F}\frac{2d\mu}{\mu}\mathcal{A}(\alpha_s(\mu)) - \mathcal{D}(\alpha_s((1-x)m))\right]\right\}.\end{aligned} \tag{3.2}$$

Here[3] $\mathcal{A}$ and $\mathcal{D}$ are Sudakov anomalous dimensions having the following perturbative expansions in the $\overline{\rm MS}$ renormalization scheme:

$$\begin{aligned}\mathcal{A}(\alpha_s(\mu)) &= \sum_{n=1}^{\infty} A_n^{\overline{\rm MS}}\left(\frac{\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^n = \frac{C_F}{\beta_0}\sum_{n=1}^{\infty} a_n^{\overline{\rm MS}}\left(\frac{\beta_0\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^n, \\ \mathcal{D}(\alpha_s(\mu)) &= \sum_{n=1}^{\infty} D_n^{\overline{\rm MS}}\left(\frac{\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^n = \frac{C_F}{\beta_0}\sum_{n=1}^{\infty} d_n^{\overline{\rm MS}}\left(\frac{\beta_0\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^n,\end{aligned} \tag{3.3}$$

[3]Notations in the literature vary; for example in ref. [29], eq. (69), our $\mathcal{A}$ and $\mathcal{D}$ are denoted by A and $-H$, respectively.

where the normalization[4] is such that $a_1 = d_1 = 1$. The coefficients $a_2^{\overline{\rm MS}}$ and $a_3^{\overline{\rm MS}}$ are known based on calculations in deep inelastic scattering (see e.g. [33]). They are given in eq. (C.3). $d_2^{\overline{\rm MS}}$ is given by:

$$d_2^{\overline{\rm MS}} = \frac{1}{9} + \frac{C_A}{\beta_0}\left(\frac{9}{4}\,\zeta_3 - \frac{\pi^2}{12} - \frac{11}{18}\right). \tag{3.4}$$

We shall explain how it was determined in section 3.5.

$\mathcal{A}(\alpha_s(\mu))$ is the universal cusp anomalous dimension which generates the double logs. It is well known that this object can be defined through the renormalization of a Wilson line with a cusp [35, 36, 37, 38, 23, 39, 40]. $\mathcal{D}(\alpha_s(\mu))$ is another anomalous dimension, which describes soft radiation from the heavy quark and it is specific to the heavy-quark distribution function. In section 3.4 we shall prove that the same anomalous dimension controls the large-N limit of the heavy-quark fragmentation function. $\mathcal{D}(\alpha_s(\mu))$ too can be defined and computed using a Wilson-line operator. In contrast with the cusp anomalous dimension, it is associated with a *specific* Π-shaped configuration with a finite light-like section to which two antiparallel infinite time-like rays attach. This object was first analyzed by Korchemsky and Marchesini in ref. [23].

The single-dressed-gluon calculation of the previous section has just leading logarithmic accuracy: eq. (2.8) (or eq. (3.10) below) fixes $C_{n,k=n+1}$ for any n, but gives only the part of the $C_{n,k\leq n}$ which is leading in β_0. In the full theory there are additional terms in the exponent which have different color factors. While $\mathcal{A}$ (the cusp anomalous dimension) contains both non-abelian C_A/β_0 and abelian C_F/β_0 terms, $\mathcal{D}$ has only non-abelian ones. These coefficients can be determined by computing the anomalous dimensions order by order. Next-to-leading logarithmic accuracy ($C_{n,k\geq n}$) requires fixing a_2 and d_1. The state of the art is next-to-next-to-leading logarithmic accuracy ($C_{n,k\geq n-1}$) which requires knowing also a_3 and d_2. Note that the calculation of the exponent to this formal accuracy should be complemented by full next-to-next-to-leading order calculations of the hard coefficient function. These are yet unavailable for radiative or semileptonic decay spectra.

Switching to the scheme invariant Borel representation of the two anomalous dimensions,

$$\begin{aligned}
\mathcal{A}(\alpha_s(\mu)) &= \frac{C_F}{\beta_0}\int_0^\infty du\, T(u)\left(\frac{\Lambda^2}{\mu^2}\right)^u B_{\mathcal{A}}(u)\,, \\
\mathcal{D}(\alpha_s(\mu)) &= \frac{C_F}{\beta_0}\int_0^\infty du\, T(u)\left(\frac{\Lambda^2}{\mu^2}\right)^u B_{\mathcal{D}}(u)\,,
\end{aligned} \tag{3.5}$$

the moments of the distribution function become:

$$F_N^{\rm PT}(m;\mu_F) = H(m;\mu_F)\, S_N(m;\mu_F) + \mathcal{O}\left(\frac{1}{N}\right), \tag{3.6}$$

[4]We shall mostly use the notation a_n and d_n for the coefficients. This is convenient for comparison with the large-β_0 limit and with the Borel formulation but it does not imply any additional approximation: a_n and d_n contain all color factors.

with[5]

$$
\begin{aligned}
S_N(m;\mu_F) &= \exp\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_s(m)}{\pi}\right)^n \sum_{k=1}^{n+1} C_{n,k}\ln^k N\right\} \\
&= \exp\left\{\frac{C_F}{\beta_0}\int_0^\infty \frac{du}{u}\,T(u)\,\left(\frac{\Lambda^2}{m^2}\right)^u \times\right. \\
&\quad \left.\times\left[B_{\mathcal{S}}(u)\Gamma(-2u)\left(N^{2u}-1\right)+\left(\frac{m^2}{\mu_F^2}\right)^u B_{\mathcal{A}}(u)\ln N\right]\right\}, \qquad (3.7)
\end{aligned}
$$

where $C_{n,k=1}$ depend on $\ln\mu_F/m$ while $C_{n,k>1}$ are numbers, and where

$$
B_{\mathcal{S}}(u) \equiv B_{\mathcal{A}}(u) - uB_{\mathcal{D}}(u)\,. \tag{3.8}
$$

These expressions are completely general.

Based on eq. (2.8) we find that the hard function that incorporates the finite terms at $N \longrightarrow \infty$ is given by:

$$
H(m;\mu_F) = 1 + \frac{C_F\alpha_s}{\pi}\left[\left(-\frac{3}{4}+\gamma_E\right)\ln\frac{m^2}{\mu_F^2} + 1 - \frac{\pi^2}{6} + \gamma_E - \gamma_E^2\right] + \mathcal{O}(\alpha_s^2)\,, \tag{3.9}
$$

and that in the large-β_0 limit the Borel transform of the anomalous dimension $B_{\mathcal{S}}(u)$ is given by

$$
B_{\mathcal{S}}(u) = \mathrm{e}^{\frac{5}{3}u}(1-u) + \mathcal{O}\left(\frac{1}{\beta_0}\right). \tag{3.10}
$$

The information contained in eqs. (3.10), (3.8) and (C.4) can readily be translated into values of the coefficients in eq. (3.3) in the large-β_0 limit. One gets:

$$
d_n^{\overline{\rm MS}}\Big|_{\text{large}\,\beta_0} = \frac{1}{n}\left[\,a_{n+1}^{\overline{\rm MS}}\Big|_{\text{large}\,\beta_0} - \left(\frac{5}{3}\right)^n\left(1-\frac{3n}{5}\right)\right]. \tag{3.11}
$$

The first few orders are summarized in table 1. Note that the convergence of the cusp anomalous dimension $\mathcal{A}$ is much faster[6] than that of $\mathcal{D}$. This property presumably persists in the full theory.

Expanding the Borel integral and the exponential in eq. (3.7) one obtains the log-enhanced terms, order by order in the coupling. In contrast with eq. (3.2), eq. (3.7) incorporates renormalon resummation in the Sudakov exponent (DGE). It exposes the fact that the exponent contains infrared renormalon ambiguities: poles in $\Gamma(-2u)$ along the integration path, which generate power-like ambiguities $\sim (N\Lambda/m)^j$, where j are integers. These parametrically-large ambiguities are indicative of corresponding non-perturbative power terms. As shown in ref. [14] (see also [39]), the $u = 1/2$ renormalon is related to the

[5] Although both eq. (3.2) and eq. (3.7) are normalized such that the $N = 1$ moment is identically unity, the integration over x in eq. (3.2) generates, in addition to the relevant $\ln N$ terms, some finite terms along with terms that vanish as powers of $1/N$. For this reason the functions in eq. (3.1) have a tilde distinguishing them from those of eq. (3.6). As far as the logarithms are concerned the exponent in eq. (3.7) is identical to that of eq. (3.2).

[6] In fact it is probably faster than all other Sudakov anomalous dimension. See e.g. table 1 in [26].

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n	$a_n^{\overline{\rm MS}}\big\vert_{{\rm large}\,\beta_0}$	$d_n^{\overline{\rm MS}}\big\vert_{{\rm large}\,\beta_0}$	ratio
1	1	1	1.
2	$\frac{5}{3}$	$\frac{1}{9}$	15.
3	$-\frac{1}{3}$	$\frac{109}{81}-\frac{2}{3}\zeta_3$	−0.6126
4	$\frac{1}{3}-2\,\zeta_3$	$\frac{1}{120}\pi^4+\frac{212}{81}-\frac{5}{6}\,\zeta_3$	−0.8531
5	$\frac{1}{30}\pi^4-\frac{1}{3}-\frac{10}{3}\zeta_3$	$\frac{6331}{1215}-\frac{6}{5}\zeta_5+\frac{1}{90}\pi^4+\frac{2}{15}\zeta_3$	−0.2099
6	$\frac{1}{3}-6\,\zeta_5+\frac{1}{18}\pi^4+\frac{2}{3}\zeta_3$	$-\frac{5}{3}\zeta_5-\frac{1}{540}\pi^4+\frac{1}{567}\pi^6+\frac{20191}{2187}+\frac{1}{3}\zeta_3^2-\frac{1}{9}\zeta_3$	0.0347

Table 1: The large-β_0 part of the coefficients $a_n^{\overline{\rm MS}}$ and $d_n^{\overline{\rm MS}}$ in eq. (3.3) and their ratios.

leading infrared renormalon in the pole mass [41, 42], while higher powers are associated with other local matrix elements which constitute the non-perturbative quark distribution function [8] or "Shape Function" [7].

In the DGE approach the Borel integration is performed using the Cauchy Principal Value prescription, making an explicit power-like separation between perturbative and non-perturbative contributions to the exponent. Any information on the anomalous dimensions from fixed-order calculations can be used in eq. (3.7). Based on eqs. (C.3) and (3.4), we have:

$$B_{\mathcal{A}}(u) = 1+\left(\frac{5}{3}+c_2\right)\frac{u}{1!}+\left(-\frac{1}{3}+c_3\right)\frac{u^2}{2!}+\mathcal{O}(u^3)\,, \tag{3.12}$$

$$B_{\mathcal{D}}(u) = 1+\left[\frac{1}{9}+\frac{C_A}{\beta_0}\left(\frac{9}{4}\zeta_3-\frac{\pi^2}{12}-\frac{11}{18}\right)\right]\frac{u}{1!}+\mathcal{O}(u^2)\,, \tag{3.13}$$

where c_n represent contributions that are subleading in β_0; c_2 and c_3 are given in appendix C. Note that the relations of c_n for $n\geq 3$ with $a_n^{\overline{\rm MS}}$ involves the coefficients of the β function. The same is true for the $\mathcal{O}(u^2)$ and higher order coefficients in $B_{\mathcal{D}}(u)$. Finally, note that since the analytic dependence on u is not known beyond the large-β_0 limit, there remains some uncertainty in evaluating the u-integral in eq. (3.7) with eqs. (3.12) and (3.13). This issue is addressed in detail in a forthcoming publication [17].

3.2 Strictly factorized form and Wilson lines

The factorization-scale dependence of the Sudakov factor of eq. (3.7) (or eq. (3.2)) is governed by the cusp anomalous dimension alone while its dependence on the soft scale m/N is different. Furthermore, by convention this factor is normalized such that its first moment ($N=1$) is unity for any μ_F; this normalization is natural since the full quark distribution function indeed obeys $F_{N=1}\equiv 1$. These constraints can be satisfied only if the exponent acquires some dependence on the hard scale m. However, such dependence cannot be consistent with the Wilson-line operator definition [35, 38, 10, 23], nor with the effective field theory approach [43, 44, 11, 12, 13]. In the latter definitions the Sudakov factor cannot depend on m, but only on the soft scale m/N and on the factorization scale, the renormalization scale of the operator. In order to convert eq. (3.7) to a strictly factorized form we reshuffle radiative corrections that depend only on the hard scales into the hard function, writing

$$F_N^{\rm PT}(m;\mu_F) = H(m;\mu_F)\, S_N(m;\mu_F) + \mathcal{O}\left(\frac{1}{N}\right) = \mathcal{H}(m;\mu_F)\, \mathcal{S}\left(\frac{N\mu_F}{m}\right) + \mathcal{O}\left(\frac{1}{N}\right) \quad (3.14)$$

where the first expression represents the definition of section 3.1 whereas the second assumes strict factorization, as in ref. [23]. Although we do not write it explicitly, both $\mathcal{H}$ and $\mathcal{S}$ depends also on $\alpha_s(\mu)$, where the renormalization scale of the coupling, μ, is not necessarily equal to the factorization scale μ_F. In practice we shall be using the standard dimensional regularization with μ defined in the $\overline{\rm MS}$ scheme for both the operator and the coupling.

We can now write the Borel representation of $\mathcal{S}(N\mu/m)$ as follows:

$$\begin{aligned} \mathcal{S}\left(\frac{N\mu}{m}\right) &= \exp\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu)}{\pi}\right)^n \sum_{k=1}^{n+1} \mathcal{C}_{n,k} \ln^k\left(\frac{N\mu}{m}\right)\right\} \\ &= \exp\left\{\frac{C_F}{\beta_0}\int_0^{\infty}\frac{du}{u}\, T(u)\left(\frac{\Lambda^2}{\mu^2}\right)^u \left[B_{\mathcal{S}}(u)\Gamma(-2u)\left(\left(\frac{N\mu}{m}\right)^{2u} - 1\right) + \right.\right. \\ &\qquad\qquad \left.\left. + B_{\mathcal{A}}(u)\ln\left(\frac{N\mu}{m}\right)\right]\right\}. \end{aligned} \quad (3.15)$$

$B_{\mathcal{S}}(u)$ and $B_{\mathcal{A}}(u)$ are the same Borel functions appearing in eq. (3.7); clearly eq. (3.15) differs from eq. (3.7) just by N-independent terms, and it is therefore consistent with eq. (3.14). Note that in contrast with eq. (3.7) the normalization (the $N = 1$ moment) of the strictly-factorized Sudakov factor of eq. (3.15) strongly depends on $\ln \mu/m$ to any order in perturbation theory.

Since there are no collinear singularities from the heavy-quark line, Sudakov logarithms in $\mathcal{S}(N\mu/m)$ are related to soft gluons and to gluons that are collinear to the lightcone direction y in eq. (2.1). One therefore expects that the log-enhanced terms of the quark distribution function eq. (2.1) would not change if the heavy-quark lines are replaced by time-like Wilson lines. The Wilson line describes the Eikonal interaction of the heavy quark with soft gluons. This is equivalent to taking the quark mass to infinity.

In order to convert eq. (2.1) to a Wilson-line operator definition [23] one applies the Eikonal approximation replacing the interacting quark field $\Psi(z)$ by a free heavy quark field $\psi(z)$ multiplied by a Wilson line in the direction p which extends to infinity, i.e.

$$\Psi(z) \longrightarrow \Phi_p(\infty, z)\psi(z)\,; \qquad \bar{\Psi}(z) \longrightarrow \bar{\psi}(z)\Phi_{-p}(z,\infty) \quad (3.16)$$

with $\Phi_p(z_1, z_2)$ defined as in appendix A. Next, one uses the fact that the free fields annihilate the external quark states to convert the matrix element to one in the vacuum. Factoring out the Dirac structure one obtains:

$$\langle h(p)|\, \bar{\Psi}(y)\Phi_y(0,y)\gamma_+\Psi(0)\, |h(p)\rangle_\mu \longrightarrow 2p^+ {\rm e}^{iy^-p^+}\, \langle 0|\, \Phi_{-p}(y,\infty)\Phi_y(0,y)\Phi_p(\infty,0)\, |0\rangle_\mu\,. \quad (3.17)$$

The $x \longrightarrow 1$ singular terms in the quark distribution function are therefore:

$$f_{\rm PT}(x;\mu)|_{x\longrightarrow 1} \sim \int_{-\infty}^{\infty}\frac{p^+dy^-}{2\pi}\, {\rm e}^{ip^+y^-(1-x)}\, \langle 0|\, \Phi_{-p}(y,\infty)\Phi_y(0,y)\Phi_p(\infty,0)\, |0\rangle_\mu\,, \quad (3.18)$$

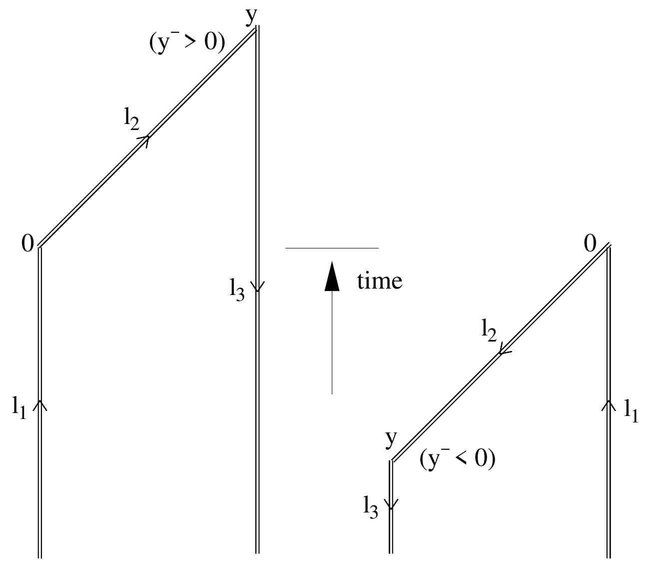

Figure 1: Minkowski space-time picture (vertical axis as time and horizontal axis as x_3) of the Wilson-line configuration $W[C_S](ip \cdot y\mu/m, \alpha_s(\mu))$ of eq. (3.19) representing the quark distribution function in an on-shell quark in the infinite mass limit (in the rest frame of this quark). The two figures describe the situation when y^- is positive (l.h.s.) or negative (r.h.s.), where path-ordering on the lightlike segment l_2 from 0 to y corresponds to time-ordering and anti-time-ordering, respectively.

Following ref. [23] we define $W[C_S]$ by

$$W[C_S]\left(\frac{ip \cdot y\mu}{m}, \alpha_s(\mu)\right) \equiv \langle 0|\, \Phi_{-p}(y,\infty)\Phi_y(0,y)\Phi_p(\infty,0)\, |0\rangle_\mu \, ; \tag{3.19}$$

it is shown in figure 1. $W[C_S](ip \cdot y\mu/m, \alpha_s(\mu))$ is an analytic function except on the negative real axis of $ip \cdot y\mu/m$. We note that, since $f_{\rm PT}(x;\mu)$ in eq. (3.18) is real, sign inversion of $p \cdot y$ is equivalent to complex conjugation, i.e.

$$W[C_S]\left(\frac{ip \cdot y\mu}{m}, \alpha_s(\mu)\right) = W[C_S]^*\left(-\frac{ip \cdot y\mu}{m}, \alpha_s(\mu)\right). \tag{3.20}$$

Up to $\mathcal{O}(1/N)$ corrections the Fourier integral in eq. (2.1) and the Mellin integral (eq. (2.6)) amount to replacing ip^+y^- by N [23, 39]. This can be shown by first extending the x integral to $-\infty$ relying on the fact that the contribution is only from the x near 1 region, then performing the x integral getting $-1/[ip^+y^- - (N-1)]$, and finally evaluating the y^- integral by residue, closing the contour through the lower half $p \cdot y$ plane. The result is

$$F_N^{\rm PT}(m;\mu) = H_W^F(m;\mu) \times W[C_S]\left(\frac{N\mu}{m}, \alpha_s(\mu)\right) + \mathcal{O}\left(\frac{1}{N}\right), \tag{3.21}$$

where $H_W^F(m;\mu)$ is a hard coefficient function. It accounts for finite terms at $N \longrightarrow \infty$ that are not captured by the Eikonal approximation of eq. (3.17). The NLO expression for this function is given in eq. (3.24) below. $W[C_S]$ is therefore related to $\mathcal{S}$ of eq. (3.15) by

$$\mathcal{S}\left(\frac{N\mu}{m}\right) \equiv \frac{W[C_S](N\mu/m, \alpha_s(\mu))}{W[C_S](1, \alpha_s(\mu))}, \tag{3.22}$$

where the additional normalization is needed to remove all N-independent pieces, which are absent in eq. (3.15) by construction.

The Wilson line operator $W[C_S]$ was analyzed in detail in ref. [23], where its evolution equation was derived and a two-loop calculation of the corresponding anomalous dimensions was performed. Here we repeated this calculation finding:

$$\begin{aligned}\ln W[C_S]\left(\frac{N\mu}{m},\alpha_s(\mu)\right) &= C_F\left[-L^2+L-\frac{5}{24}\pi^2\right]\frac{\alpha_s(\mu)}{\pi}+\\ &+C_F\left[\left(-\frac{11\,C_A}{18}+\frac{N_f}{9}\right)L^3+\left(\left(-\frac{17}{18}+\frac{\pi^2}{12}\right)C_A+\frac{N_f}{9}\right)L^2+\right.\\ &+\left(\left(-\frac{55}{108}+\frac{9}{4}\zeta_3-\frac{7\,\pi^2}{18}\right)C_A+\left(-\frac{1}{54}+\frac{\pi^2}{18}\right)N_f\right)L+\\ &\left.+\,\mathcal{O}(1)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2+\cdots,\end{aligned}\tag{3.23}$$

where $L=\ln N\tilde{\mu}/m$ with $\tilde{\mu}=\mu\,\mathrm{e}^{\gamma_E}$ where μ is defined in the $\overline{\mathrm{MS}}$ scheme.

The *same* logarithmic terms can be obtained from eq. (3.15) by expressing it in terms of L,

$$\ln\mathcal{S}\left(\frac{N\mu}{m}\right)=\frac{C_F}{\beta_0}\int_0^\infty\frac{du}{u}\,T(u)\left(\frac{\Lambda^2}{\mu^2}\right)^u\left[B_{\mathcal{S}}(u)\Gamma(-2u)\left(\mathrm{e}^{2(L-\gamma_E)u}-1\right)+B_{\mathcal{A}}(u)(L-\gamma_E)\right],$$

substituting the anomalous dimension coefficients of eqs. (3.8), (3.12) and (3.13) and expanding. The N-independent terms, such as $-\frac{5}{24}\pi^2C_F\alpha_s(\mu)/\pi$, are of course different, see eq. (3.22). For $H_W^F(m;\mu)$ in eq. (3.21) we find:

$$H_W^F(m;\mu)=1+C_F\frac{\alpha_s(\mu)}{\pi}\left[\ln^2\frac{m}{\mu}-\frac{1}{2}\ln\frac{m}{\mu}+\frac{\pi^2}{24}+1\right]+\cdots.\tag{3.24}$$

Our result in eq. (3.23) agrees with that of ref. [23] except for the rational number in the coefficient of C_A in the single log term at order α_s^2, where we find $-55/108$ rather than $-37/108$ appearing in eq. (3.6) of ref. [23]. This discrepancy is due to an error in ref. [23] in the evaluation of one of the diagrams — see section 3.5 and appendix D below.

3.3 Evolution equation

From eq. (3.15) above one can be obtain an evolution equation by taking a full logarithmic derivative with respect to the scale. We obtain:

$$\begin{aligned}\frac{\partial\ln\mathcal{S}(N\mu/m)}{\partial\ln\mu}+\frac{\partial\ln\mathcal{S}(N\mu/m)}{\partial\alpha_s(\mu)}\frac{d\alpha_s(\mu)}{d\ln\mu}&=\\ \frac{d\ln\mathcal{S}(N\mu/m)}{\ln\mu}&=-2\mathcal{A}(\alpha_s(\mu))\,\ln\frac{N\tilde{\mu}}{m}+\Gamma_{\mathcal{S}}(\alpha_s(\mu))\end{aligned}\tag{3.25}$$

where, as above, $\tilde{\mu}=\mu\,\mathrm{e}^{\gamma_E}$, and the anomalous dimension $\Gamma_{\mathcal{S}}$ can be related order by order to the two anomalous dimensions $\mathcal{D}$ and $\mathcal{A}$ defined above:

$$\Gamma_{\mathcal{S}}(\alpha_s(\mu))=\frac{C_F}{\beta_0}\int_0^\infty du T(u)\left(\frac{\Lambda^2}{\mu^2}\right)^u\left[\Gamma(1-2u)B_{\mathcal{D}}(u)+\frac{1+2u\gamma_E-\Gamma(1-2u)}{u}B_{\mathcal{A}}(u)\right].\tag{3.26}$$

For its expansion we obtain:

$$\Gamma_{\mathcal{S}}(\alpha_s(\mu)) = C_F\left\{\frac{\alpha_s(\mu)}{\pi}+\beta_0\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left[d_2^{\overline{\rm MS}}-\frac{\pi^2}{3}+2\gamma_E-2\gamma_E^2\right]+\cdots\right\} \tag{3.27}$$

$$= C_F\left\{\frac{\alpha_s(\mu)}{\pi}+\beta_0\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left[\frac{1}{9}-\frac{\pi^2}{3}+2\gamma_E-2\gamma_E^2+\frac{C_A}{\beta_0}\left(\frac{9}{4}\zeta_3-\frac{\pi^2}{12}-\frac{11}{18}\right)\right]\right\}.$$

Note that eq. (3.25) is similar but not identical to the equation derived in ref. [23] for $W[C_S]$ (eq. (4.5) in ref. [23][7]). The equation for $W[C_S]$ takes the form

$$\frac{d\ln W[C_S](N\mu/m,\alpha_s(\mu))}{d\ln\mu} = -2\mathcal{A}(\alpha_s(\mu))\ln\frac{N\tilde{\mu}}{m}-\Gamma(\alpha_s(\mu))\,, \tag{3.28}$$

where, according to eq. (3.22),

$$\Gamma(\alpha_s(\mu)) = -\Gamma_{\mathcal{S}}(\alpha_s(\mu))+\frac{\partial\ln W[C_S](1,\alpha_s(\mu))}{\partial\alpha_s}\frac{d\alpha_s}{d\ln\mu}.$$

From eq. (3.23) we have

$$\ln W[C_S](1,\alpha_s(\mu)) = C_F\frac{\alpha_s(\mu)}{\pi}\left(-\frac{5}{24}\pi^2+\gamma_E-\gamma_E^2\right)+\mathcal{O}(\alpha_s^2)\,,$$

so the NLO result for $\Gamma(\alpha_s(\mu))$ is:

$$\Gamma(\alpha_s(\mu)) = -C_F\left\{\frac{\alpha_s(\mu)}{\pi}+\beta_0\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left[\frac{1}{9}+\frac{\pi^2}{12}+\frac{C_A}{\beta_0}\left(\frac{9}{4}\zeta_3-\frac{\pi^2}{12}-\frac{11}{18}\right)\right]\right\} \tag{3.29}$$

$$= -C_F\frac{\alpha_s(\mu)}{\pi}+C_F\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left[\left(\frac{1}{54}+\frac{\pi^2}{72}\right)N_f+\left(\frac{55}{108}+\frac{\pi^2}{144}-\frac{9}{4}\zeta_3\right)C_A\right].$$

3.4 Relation between distribution and fragmentation in the Sudakov region

In the following we show that the function controlling the large-N limit of the perturbative heavy-quark *fragmentation function* is identical to $\mathcal{S}(N\mu/m)$ of the heavy-quark *distribution function*.

According to the discussion in section 3.2 above, the log-enhanced terms in the distribution function can be computed in the Eikonal approximation, namely using the Wilson line definition of eq. (3.19). Following section 2.3 in ref. [23] we now show that the same applies in the case of the fragmentation function. We then prove that in the Sudakov limit the two objects are in fact identical. This is a new result.

The fragmentation function is defined by [4] (see eq. (5.2) there):

$$d(x;\mu) \equiv \frac{x^{1-2\epsilon}}{2\pi}\int_{-\infty}^{\infty}dy_-\exp\left(-\frac{ip^+y^-}{x}\right)\times \tag{3.30}$$

$$\times\frac{1}{4N_c}\,{\rm Tr}\left\{\sum_X\gamma_+\langle 0|\Psi(0)\Phi^*_{-y}(\infty,0)|H(p)+X\rangle\langle H(p)+X|\Phi^*_y(y,\infty)\overline{\Psi}(y)|0\rangle_\mu\right\},$$

[7] $\Gamma(g)$ quoted below eq. (4.5) in ref. [23] is inconsistent with the two-loop result obtained in section 3 of the same paper. This inconsistency was independently identified by Neubert [46].

where dimensional regularization in $D = 4 - 2\epsilon$ dimensions is assumed and μ is the renormalization scale of the operator. Here the trace is taken over Dirac and color indices, the sum is over all hadronic states X that can be produced together with the observed heavy hadron $H(p)$, p is the momentum of the latter which is assumed to have no transverse component and y is a lightlike vector in the "$-$" direction. For the Wilson lines we use the notation of eqs. (A.1) and (A.2), where * stands for complex conjugation; ref. [4] expresses the same Wilson line in terms of transposed color matrices, i.e.

$$\Phi^*_{-y}(\infty, 0) = \left[\overline{\mathbf{P}} \exp\left(-ig \int_0^\infty dy^- A_a^+(y^-) t_a\right)\right]^* = \overline{\mathbf{P}} \exp\left(ig \int_0^\infty dy^- A_a^+(y^-) t_a^T\right).$$

It is straightforward to check that $\Psi(0)\, \Phi^*_{-y}(\infty, 0)$ is gauge invariant if the gauge field does not transform at infinity. Since $d(x;\mu)$ has support in $x \in [0,1]$ one can define moments as usual,

$$D_N(m;\mu) = \int_0^1 dx x^{N-1} d(x;\mu)\,. \tag{3.31}$$

In the perturbative analogue of $d(x;\mu)$, which we denote by $d_{\rm PT}(x;\mu)$, $H(p)$ is replaced by an on-shell heavy quark $h(p)$ with $p^2 = m^2$. Contrary to the non-perturbative definition, in perturbation theory $h(p)$ carries color, and so does X.

Because there are no collinear singularities from the heavy-quark propagator, Sudakov logarithms are either due to soft gluons, or to gluons that are collinear with the "$-$" direction. Therefore, as argued in ref. [23], the Sudakov limit can be safely studied in the approximation where the dynamical heavy-quark field in each amplitude is replaced by a free quark multiplied by a Wilson line along the quark trajectory:

$$\Psi(z) \longrightarrow \psi(z)\, \Phi^*_p(z, \infty)\,; \qquad \bar{\Psi}(z) \longrightarrow \Phi^*_{-p}(\infty, z)\, \bar{\psi}(z)\,, \tag{3.32}$$

where the path-ordered exponential is defined as in eq. (A.1).

Taking the infinite-mass limit implies that the detected on-shell quark $h(p)$ is produced from the heavy-quark field in the operator rather than from some vacuum fluctuation. Note that when considering the Sudakov limit such fluctuations are irrelevant *even* if the mass is not large, since they necessarily involve gluon splitting into a quark-antiquark pair, which is regular at $x \longrightarrow 1$. Therefore, in this approximation the free quark field ψ annihilates $h(p)$ from the external states. Factoring out the Dirac structure one obtains:

$$\begin{aligned}
&\frac{1}{4N_c} \mathrm{Tr} \left\{ \sum_X \gamma_+ \langle 0|\Psi(0) \Phi^*_{-y}(\infty, 0)|h(p) + X\rangle\langle h(p) + X|\Phi^*_y(y, \infty)\overline{\Psi}(y)|0\rangle_\mu \right\} \longrightarrow \\
&\quad \longrightarrow p^+ \,\mathrm{e}^{ip^+y^-} \langle 0|\Phi^*_p(0, \infty)\, \Phi^*_{-y}(y, 0)\Phi^*_{-y}(\infty, y) \sum_X |X\rangle\langle X|\Phi^*_y(y, \infty)\Phi^*_{-p}(\infty, y)|0\rangle_\mu \\
&\quad = p^+ \,\mathrm{e}^{ip^+y^-} \Big[\langle 0|\Phi_p(0, \infty)\Phi_{-y}(y, 0)\, \Phi_{-p}(\infty, y)|0\rangle_\mu\Big]^*\,,
\end{aligned} \tag{3.33}$$

where we relied on completeness of the set of states $|X\rangle$ (which close the color trace) and on the properties of the Wilson lines in eq. (A.3). We therefore find that the $x \longrightarrow 1$

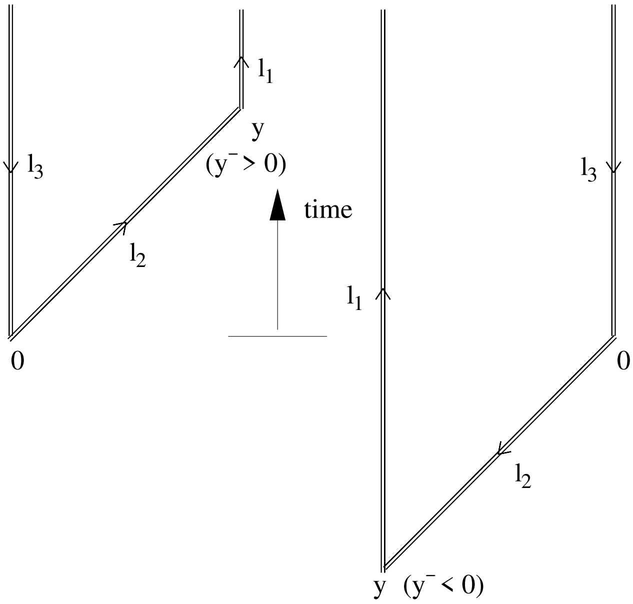

Figure 2: Minkowski space-time picture of the Wilson-line configuration in the second line of eq. (3.34), i.e. $W^*[C_S](ip\cdot y\mu/m, \alpha_s(\mu)) = W[C_S](-ip\cdot y\mu/m, \alpha_s(\mu))$, representing the perturbative fragmentation function in the infinite-mass limit (in the rest frame of the produced quark). The two figures describe the situation when y^- is positive (l.h.s.) or negative (r.h.s.), where path ordering on the lightlike segment l_2 from 0 to y corresponds to time-ordering and anti-time-ordering, respectively; cf. figure 1.

singular terms in the fragmentation function are summarized by:

$$\begin{aligned} d_{\rm PT}(x;\mu)|_{x\longrightarrow 1} &\sim \int_{-\infty}^{\infty} \frac{p^+dy^-}{2\pi}\, {\rm e}^{-ip^+y^-(1-x)}\, \langle 0|\, \Phi_p(0,\infty)\, \Phi_{-y}(y,0)\, \Phi_{-p}(\infty,y)\, |0\rangle_\mu^\dagger \\ &= \int_{-\infty}^{\infty} \frac{p^+dy^-}{2\pi}\, {\rm e}^{-ip^+y^-(1-x)}\, \langle 0|\, \Phi_p(y,\infty)\, \Phi_{y}(0,y)\, \Phi_{-p}(\infty,0)\, |0\rangle_\mu \\ &= \int_{-\infty}^{\infty} \frac{-q^+dy^-}{2\pi}\, {\rm e}^{iq^+y^-(1-x)}\, \langle 0|\, \Phi_{-q}(y,\infty)\, \Phi_{y}(0,y)\, \Phi_{q}(\infty,0)\, |0\rangle_\mu \\ &= \int_{-\infty}^{\infty} \frac{-q^+dy^-}{2\pi}\, {\rm e}^{iq^+y^-(1-x)}\, W[C_S]\bigg(\frac{iq\cdot y\mu}{m}, \alpha_s(\mu)\bigg) \\ &= \int_{-\infty}^{\infty} \frac{p^+dy^-}{2\pi}\, {\rm e}^{-ip^+y^-(1-x)}\, W[C_S]\bigg(-\frac{ip\cdot y\mu}{m}, \alpha_s(\mu)\bigg)\,, \end{aligned} \tag{3.34}$$

where in the third line we defined $q \equiv -p$ and in the fourth we identified the matrix element as $W[C_S]$ that was defined in the context of the *distribution function*, eq. (3.19) above. Finally, in the last line we returned to the original variable p finding that the function $W[C_S]$ is evaluated at $-ip\cdot y\mu/m$. Since $d_{\rm PT}(x;\mu)$ is real, sign inversion of the argument of $W[C_S]$ is equivalent to its complex conjugation (cf. eq. (3.20)) so we get

$$d_{\rm PT}(x;\mu)|_{x\longrightarrow 1} \sim \int_{-\infty}^{\infty} \frac{p^+dy^-}{2\pi}\, {\rm e}^{ip^+y^-(1-x)}\, W[C_S]\bigg(\frac{ip\cdot y\mu}{m}, \alpha_s(\mu)\bigg)\,, \tag{3.35}$$

the same expression as for $f_{\mathrm{PT}}(x;\mu)$ in eq. (3.18). Therefore, we find that as far as the $x \longrightarrow 1$ terms are concerned, $d_{\mathrm{PT}}(x;\mu)$ is identical to $f_{\mathrm{PT}}(x;\mu)$. Having the relation in the second line eq. (3.34) above, ref. [23] has defined

$$W[C_T]\left(\frac{ip\cdot y\mu}{m}, \alpha_s(\mu)\right) \equiv \langle 0|\, \Phi_p(y,\infty)\, \Phi_y(0,y)\, \Phi_{-p}(\infty,0)\, |0\rangle_\mu\,. \tag{3.36}$$

Here we find that $W[C_T]$ is related to $W[C_S]$ by complex conjugation:

$$W[C_T]\left(\frac{ip\cdot y\mu}{m}, \alpha_s(\mu)\right) = W[C_S]^*\left(\frac{ip\cdot y\mu}{m}, \alpha_s(\mu)\right). \tag{3.37}$$

Finally, converting eq. (3.35) to moment space $ip^+y^- \longrightarrow N$ (see eq. (9) in ref. [30]):

$$D_N^{\mathrm{PT}}(m;\mu) = H_W^D(m;\mu)\, \times\, W[C_S]\left(\frac{N\mu}{m}, \alpha_s(\mu)\right) + \mathcal{O}\left(\frac{1}{N}\right), \tag{3.38}$$

so one obtains the *same Sudakov exponent as in the distribution function to any logarithmic accuracy.* The N-independent terms are summarized by $H_W^D(m;\mu)$. At $\mathcal{O}(\alpha_s)$ $H_W^D(m;\mu)$ is equal to $H_W^F(m;\mu)$ of eq. (3.24), but this may not persist at higher orders.

3.5 Comments on the calculation of the Sudakov exponent to NNLO

In this section we explain how the two-loop coefficient $d_2^{\overline{\mathrm{MS}}}$ of eq. (3.4) was obtained. Having established the all-order equality between the Sudakov exponents of the heavy quark distribution and fragmentation functions as well as the relation between the definition based on a dynamical heavy quark with a finite on-shell mass and the Wilson-line definition, there are several way to proceed. We follow two:

- Perform a two-loop calculation using Wilson lines, as done in ref. [23] by Korchemsky and Marchesini.
- Extract the non-abelian[8] $N \longrightarrow \infty$ singular terms from a recent result for the fragmentation function by Melnikov and Mitov [31], which uses dynamical heavy quarks with a finite mass.

Beginning with the latter, the two-loop calculation of ref. [31] conveniently suites our purpose, as it relies on a *process-independent definition* of the perturbative fragmentation function in dimensional regularization [29], where process dependent power corrections in the hard scale (the scale at which the heavy quark is produced) are avoided by taking the *quasi-collinear limit* [45, 29]. In this limit the gluon transverse momentum and the quark mass are taken *small* while the ratio between them, which depends on the quark longitudinal momentum fraction, is fixed. Ref. [29] established this definition and applied it at $\mathcal{O}(\alpha_s)$, conforming previous result [27] which was obtained from heavy-quark production cross-section in e^+e^- annihilation. Ref. [29] also presented results for Sudakov resummation to NLL accuracy. Ref. [30] extended the process-independent calculation of

[8]The $C_F N_f$ term was known already in ref. [30], and was confirmed by ref. [31].

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the fragmentation function to all orders in the large-β_0 limit, which, in particular, fixes the abelian part of $d_2^{\overline{\rm MS}}$, see eq. (3.11) and table 1 above.

Ref. [31] gives a general two-loop result for $d_{\rm PT}(x;\mu)$ in momentum fraction space. It also summarizes in moment space the terms which are non-vanishing at large-N in eq. (65). We note[9] that the coupling in this paper is renormalized assuming N_f+1 dynamical massless quarks, where the additional flavor corresponds to the heavy quark. This gives rise to $C_F T_R \alpha_s^2$ terms which are not accompanied by N_f. Converting to our definition of the coupling, with N_f light quarks, these contributions drop out. The remaining terms in eq. (65) of ref. [31] match the general expression of eq. (3.6) with eqs. (3.7) and (3.9) above *provided* that $d_2^{\overline{\rm MS}}$ is given by eq. (3.4).

Here we performed[10] a two-loop calculation of $W[C_S]$ of eq. (3.19), along the lines of ref. [23]. We recall that $W[C_S]$ is a path-ordered exponential with two antiparallel rays in the timelike directions n_μ and $-n_\mu$, which are connected by a finite lightlike segment y_-. The timelike direction n_μ is determines by the heavy-quark momentum: $p_\mu = mn_\mu$, *but* in contrast with the original definitions, eqs. (2.1) and (3.30), and with the calculation of ref. [31] discussed above, the heavy quark is no more a dynamic field. It is replaced by a Wilson line that represents interaction with soft gluons only (collinear singularities do arise though from the lightlike segment). As shown in the previous sections $W[C_S]$ captures the log-enhanced terms in the heavy quark distribution and fragmentation functions to all orders, i.e. to any logarithmic accuracy. Working in $D=4-2\epsilon$ dimensions in configuration space with the Feynman rules as in appendix A of ref. [23] the standard $\overline{\rm MS}$ scale μ is introduced by making the following replacements:

$$\begin{aligned} n\cdot y &\longrightarrow n\cdot y\,\mu\,,\\ \frac{g^2}{4\pi^2} &\longrightarrow \frac{\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\left(\frac{{\rm e}^{\gamma_E}}{4\pi}\right)^{\epsilon}\left(1-\frac{\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\frac{\beta_0}{\epsilon}+\mathcal{O}(\alpha_s^2)\right). \end{aligned} \tag{3.39}$$

Our final result for the renormalized $\ln W[C_S]$ to $\mathcal{O}(\alpha_s^2)$ is summarized by eq. (3.23) above. Given that the description of the calculation in ref. [23] is clear and detailed, we shall not repeat it here. Instead, we give a full account of the calculation of one diagram, diagram 11, the one by which we differ from ref. [23]. This is done in appendix D.

4. Conclusions

In this paper we studied perturbative aspects of the heavy-quark distribution function, which has a central role in precision phenomenology of inclusive B-decay spectra, primarily in $\bar{B}\longrightarrow X_s\gamma$ and semileptonic decays. Our results include all-order resummation of running-coupling effects, in eq. (2.9), as well as determination of the Sudakov exponent in eq. (3.2) to two-loop order. The result for $d_2^{\overline{\rm MS}}$ is eq. (3.4) is now established in two

[9] I wish to thank Matteo Cacciari and Kiril Melnikov for related discussions.

[10] As mentioned following eq. (3.23), the result quoted in ref. [23] differs from my own in the non-abelian coefficient of the single log term, the one that determines d_2. Historically, it was the discrepancy with d_2 I extracted from ref. [31] which convinced me to repeat the two-loop calculation of ref. [23]. I wish to thank Gregory Korchemsky for his encouragement.

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entirely different calculation procedures: the one performed here following ref. [23] using a Wilson-line operator in configuration space and the one of ref. [31] for the fragmentation function, using the quasi-collinear limit [29, 45] in momentum space. With the jet-function anomalous dimension already being known to this order (see e.g. [26]), the Sudakov exponent in inclusive decay spectra is now determined to the NNLL accuracy. As usual, this should be matched by the computation of N-independent terms at $\mathcal{O}(\alpha_s^2)$, which are not yet available.

In addition to presenting the result for Sudakov resummation in the conventional way (eq. (3.2)) that suites fixed-logarithmic-accuracy calculations, following previous work on DGE [14, 15, 16, 30] we formulated the resummation in eq. (3.7) as a scheme-invariant Borel sum, where power-like separation between perturbative and non-perturbative contributions can be implemented by taking a Principal Value prescription. This approach is particularly advantageous for inclusive B-decay spectra where non-perturbative corrections are substantial. The application of this approach to phenomenology is already under way [17].

The quark distribution function is defined here assuming a finite on-shell quark mass, while much of the literature on inclusive decays is based on defining it in the $m \longrightarrow \infty$ limit. The evolution properties of these objects are different. Starting in section 3.2 with the QCD Sudakov resummation formula for the case of a finite on-shell mass we derived a strictly factorized form that is consistent with the $m \longrightarrow \infty$ limit. In the former the Sudakov factor is naturally defined with factorization-scale independent normalization. This is realized in eq. (3.2) and in eq. (3.7). This is not the case in the strictly-factorized formula of eq. (3.15), where the first moment strongly depends on the scale. Strict factorization (eq. (3.14)) *implies* that both the Sudakov factor $\mathcal{S}$ and the hard factor $\mathcal{H}$ acquire *double logarithmic* dependence on the scale to any order in perturbation theory (see e.g. eq. (3.24)). Therefore, in this case *both* need to be resummed.

The most interesting finding of our present investigation is the similarity of the distribution and fragmentation functions. First we found that in the large-β_0 limit these functions are identical. Then, we showed that in the Sudakov limit they are represented by the *same* Wilson-line operator — see eq. (3.34) — so the *Sudakov exponent in the two cases is identical to all orders.* We emphasize that the diagrammatic realization of this relation is non trivial: upon calculating separately virtual corrections to the fragmentation process one encounters additional Coulomb-phase contributions in individual diagrams, that are absent in the distribution case. Our result implies that these contributions cancel out in the sum of all diagrams so they make no effect on the Sudakov exponent.

In spite of these strong relations the distribution and fragmentation functions, defined in a process-independent way in dimensional regularization, are not equal. Their DGLAP evolution away from the large-x limit starts differing already at two-loop order. Additional differences between these functions appear when considering heavy quark-antiquark pairs that were neglected here; these are important in the case of fragmentation for low masses and away from the large-x limit. In spite of the similarity in the renormalon structure it is hard to imagine that there is any relation between power corrections in the two cases. Recall that the distribution function is defined with a single hadron in the initial state and a completely inclusive final state, making it a forward hadronic matrix element. On the

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other hand in the case of fragmentation both the detected hadron and the jet are in the final state and they interact by exchanging soft gluons throughout the hadronization process.

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A. Properties of Wilson lines

We define the Wilson-line operator by

$$\Phi_p(z_1,z_2) \equiv \mathbf{P}\exp\left(ig\int_{z_1}^{z_2} dz_\mu A^\mu(z)\right). \tag{A.1}$$

with $A^\mu(z) = A_a^\mu(z)t_a$ where t_a are $\mathrm{SU}(N_c)$ generators ($t_a^\dagger = t_a$) in the fundamental representation and $\mathbf{P}\exp$ indicates that matrices and fields are path-ordered. The notation $\Phi_p(z_1,z_2)$ assumes[11] that the direction p is from z_1 to z_2. Anti-path-ordering is denoted by:

$$\Phi_{-p}(z_2,z_1) \equiv \overline{\mathbf{P}}\exp\left(-ig\int_{z_1}^{z_2} dz_\mu A^\mu(z)\right). \tag{A.2}$$

The following properties of $\Phi_p(z_1,z_2)$ are useful:

$$\begin{array}{lrcll} \text{causality} & \Phi_p(z_2,z_3)\Phi_p(z_1,z_2) &=& \Phi_p(z_1,z_3) & \text{where } z_2 \text{ is between } z_1 \text{ and } z_3 \\ \text{hermiticity} & \Phi_p^\dagger(z_1,z_2) &=& \Phi_{-p}(z_2,z_1) & \\ \text{unitarity} & \Phi_p^\dagger(z_1,z_2)\Phi_p(z_1,z_2) &=& 1\,. & \end{array} \tag{A.3}$$

B. One-loop integrals with a Borel-modified propagator

Let us compute the quark distribution on an on-shell heavy quark with a single dressed gluon based the definition of eq. (2.1). First we note that the numerator is identical to the one computed in section 3.1 of ref. [30] with the replacement of $\mathcal{M}_0\bar{\mathcal{M}}_0$ by $\not{n}$, where n is in the "$-$" direction. Upon using a Borel-modified gluon propagator as well as dimensional regularization we get:

$$\begin{aligned} f_{\mathrm{PT}}(x;\mu)|_{\mathcal{O}(\alpha_s)} &= \int_{-\infty}^{\infty}\frac{dy^-}{4\pi}\,\mathrm{e}^{-ixp^+y^-}\,\left\langle h(p)\right|\bar{\Psi}(y)\Phi_y(0,y)\gamma_+\Psi(0)\left|h(p)\right\rangle_\mu \\ &= \frac{-i4C_Fg^2}{\pi}\,p^+\int_{-\infty}^{\infty}\frac{dy^-}{4\pi}\,\mathrm{e}^{-ixp^+y^-}\int\frac{d^Dk}{(2\pi)^D}\mathrm{e}^{iy^-(p^++k^+)}\times \\ &\quad\times\left[\frac{\frac{k^+}{p^+}+2+2\frac{p^+}{k^+}}{(-k^2)^{1+u}((p+k)^2-m^2)}-\frac{(2m^2+k^2)(1+\frac{k^+}{p^+})}{(-k^2)^{1+u}((p+k)^2-m^2)^2}\right], \end{aligned} \tag{B.1}$$

where the k^+/p^+ part in the first term and the entire second term originate in the Feynman gauge part of the gluon propagator and the rest is specific to the Axial gauge (2.2).

[11] There is some redundancy in the notation as $z_2 - z_1$ is parallel to p, but it is, nevertheless, convenient.

To perform the momentum integration we use the fact that under the y^- integral, a factor of $-k^+/p^+$ in the numerator becomes $(1-x)$ while the inverse factor becomes $1/(1-x)$. We therefore have to deal with just one type of integral:

$$I(m^2,x;a,b,D) \equiv p^+ \int_{-\infty}^{\infty} \frac{dy^-}{2\pi}\, \mathrm{e}^{-ixp^+y^-} \int \frac{d^Dk}{(2\pi)^D} \mathrm{e}^{iy^-(p^+ + k^+)} \frac{1}{(-k^2)^a((p+k)^2-m^2)^b}\,. \tag{B.2}$$

Using Feynman parametrization and shifting the integration momentum $q = k + p(1-\alpha)$ and changing the order of integration we obtain:

$$\begin{aligned}
I(m^2,x;a,b,D) &= p^+ \frac{(-1)^{-a}\Gamma(a+b)}{\Gamma(b)\Gamma(a)} \int_0^1 d\alpha \alpha^{a-1}(1-\alpha)^{b-1} \int_{-\infty}^{\infty} \frac{dy^-}{2\pi}\, \mathrm{e}^{-i(x-\alpha)p^+y^-} \times \\
&\quad \times \int \frac{d^Dq}{(2\pi)^D} \mathrm{e}^{iy^-q^+} \frac{1}{(q^2-m^2(1-\alpha)^2)^{a+b}} \\
&= \frac{i(-1)^b\Gamma(a+b-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(b)\Gamma(a)} \int_0^1 d\alpha \delta(\alpha - x) \alpha^{a-1}(1-\alpha)^{b-1}(m^2(1-\alpha)^2)^{\frac{D}{2}-a-b} \\
&= \frac{i(-1)^b\Gamma(a+b-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(b)\Gamma(a)} (m^2)^{\frac{D}{2}-a-b} x^{a-1}(1-x)^{D-1-2a-b}\,,
\end{aligned} \tag{B.3}$$

where we first performed the momentum integration (observing that the exponential $\mathrm{e}^{iy^-q^+}$ can be replaced by 1 since the result is a scalar). Then we performed the y^- integration getting a Dirac $\delta(\alpha - x)$. This made the integration over the Feynman parameter α trivial. We comment that had we used the Eikonal approximation for the massive propagator, i.e. $(p+k)^2 - m^2 \longrightarrow 2pk$ we would have obtained the same answer (same Γ functions) but the dependence on x would have been modified such that $x^{a-1} \longrightarrow 1$, not affecting the large-x limit.

Eq. (B.1) can now be computed by the appropriate assignments in eq. (B.3). The result is given by eq. (2.3).

C. The splitting function and the cusp anomalous dimension

The non-singlet splitting function has been recently computed to three loops by Moch, Vermaseren and Vogt [33]. In addition, its large-β_0 limit is known to all orders and it is given by [34]

$$\begin{aligned}
\gamma(N,a) &= \frac{C_F}{\beta_0} \sum_{n=0}^{\infty} \gamma_n(N) a^{n+1} \\
&= \mathcal{A}\Bigg[\Psi(N+a) - \Psi(1+a) + \\
&\quad + \frac{N-1}{2}\left(\frac{a^2+2a-1}{1+a}\frac{1}{N+a} - \frac{(1+a)^2}{2+a}\frac{1}{N+1+a}\right)\Bigg] + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)
\end{aligned} \tag{C.1}$$

where $\mathcal{A}$ is the large-β_0 limit of the cusp anomalous dimension [36, 35, 38, 23, 40] discussed below, and $a \equiv a(\mu) = \beta_0\alpha_s(\mu)/\pi$ is the large-β_0 coupling in $\overline{\mathrm{MS}}$.

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The perturbative expansion of the cusp anomalous dimension (the anomalous dimension of an operator made of two Wilson lines with a cusp), which is also the large-N limit of the quark-quark splitting function, is given by

$$\begin{aligned}\mathcal{A}(\alpha_s(\mu)) &= \frac{C_F}{\beta_0}\int_0^\infty du\, T(u)\left(\frac{\Lambda^2}{\mu^2}\right)^u B_{\mathcal{A}}(u) \\ &= \frac{C_F}{\beta_0}\left[\left(\frac{\beta_0\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right) + a_2^{\overline{\rm MS}}\left(\frac{\beta_0\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^2 + a_3^{\overline{\rm MS}}\left(\frac{\beta_0\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^3 + \cdots\right],\end{aligned} \tag{C.2}$$

where the coefficient $a_3^{\overline{\rm MS}}$ is known from the recent calculation of the splitting function [33]. Explicitly,

$$\begin{aligned}a_2^{\overline{\rm MS}} &= \frac{5}{3} + \frac{C_A}{\beta_0}\left(\frac{1}{3} - \frac{\pi^2}{12}\right), \\ a_3^{\overline{\rm MS}} &= -\frac{1}{3} + \frac{1}{\beta_0}\left[\left(\frac{55}{16} - 3\,\zeta_3\right) C_F + \left(\frac{253}{72} - \frac{5\,\pi^2}{18} + \frac{7}{2}\zeta_3\right) C_A\right] + \\ &\quad + \frac{1}{{\beta_0}^2}\left[\left(-\frac{605}{192} + \frac{11}{4}\zeta_3\right) C_A\, C_F + \left(-\frac{7}{18} - \frac{\pi^2}{18} - \frac{11}{4}\zeta_3 + \frac{11\,\pi^4}{720}\right) {C_A}^2\right].\end{aligned} \tag{C.3}$$

In the large-β_0 limit [34, 40]:

$$\mathcal{A}(\alpha_s(\mu)) = \frac{C_F}{\beta_0}\frac{\sin\pi a}{\pi}\frac{\Gamma(4+2a)}{6\Gamma(2+a)^2} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right). \tag{C.4}$$

The Borel representation of $\mathcal{A}(\alpha_s(\mu))$ in the full theory can be written in an expanded form as in eq. (3.12), where c_n represent the terms that are subleading in β_0. Upon comparing the latter with the second line in eq. (C.2) and using eq. (2.5) we get:

$$\begin{aligned}c_2 &= a_2^{\overline{\rm MS}} - \frac{5}{3} \\ c_3 &= a_3^{\overline{\rm MS}} + \frac{1}{3} + \delta_2^{\overline{\rm MS}} - \delta a_2^{\overline{\rm MS}},\end{aligned} \tag{C.5}$$

where the term involving $\delta_2 = \beta_2^{\overline{\rm MS}}/\beta_0^3$ is due to converting from $\overline{\rm MS}$ to the 't Hooft scheme (see eq. (27) in [26]). Explicitly this gives:

$$\begin{aligned}c_2 &= \frac{C_A}{\beta_0}\left(\frac{1}{3} - \frac{\pi^2}{12}\right) \\ c_3 &= \frac{1}{\beta_0}\left[\left(\frac{649}{288} - \frac{5}{18}\pi^2 + \frac{7}{2}\zeta\,(3)\right) C_A + \left(\frac{23}{8} - 3\zeta_3\right) C_F\right] + \\ &\quad + \frac{1}{\beta_0^2}\left[\left(\frac{251}{288} + \frac{7}{144}\pi^2 - \frac{11}{4}\zeta_3 + \frac{11}{720}\pi^4\right) C_A^2 + \right. \\ &\qquad \left. + \left(-\frac{235}{96} + \frac{11}{4}\zeta_3 + \frac{\pi^2}{16}\right) C_F C_A - \frac{3}{32}C_F^2\right] + \\ &\quad + \frac{1}{\beta_0^3}\left[\left(-\frac{301}{512} - \frac{7}{192}\pi^2\right) C_A^3 + \left(-\frac{11}{64} - \frac{11}{192}\pi^2\right) C_F C_A^2 + \frac{11}{128} C_F^2 C_A\right].\end{aligned} \tag{C.6}$$

The presence of a $\mathcal{O}(1/\beta_0^3)$ term is, of course, a special feature of our specific Borel representation (or of the 't Hooft scheme).

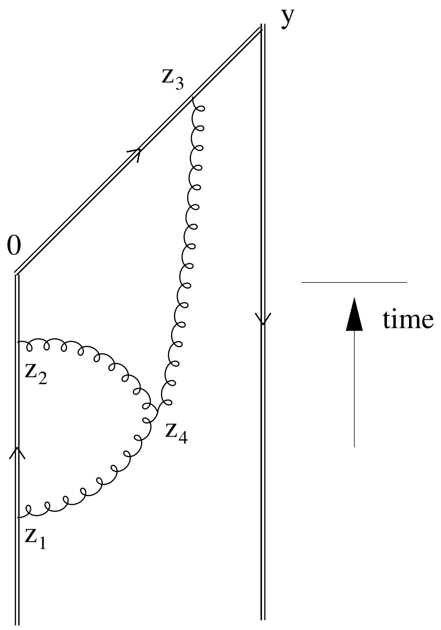

Figure 3: One of the non-abelian two-loop diagrams contributing to $\ln W[C_S]$ of eq. (3.19). It corresponds to diagram 11 in figure 6 of ref. [23] (published version).

D. Calculation of diagram 11 in ref. [23]

Let us focus here on the calculation of the non-abelian two-loop diagram shown in figure 3 in the Feynman gauge. The calculation is done in configuration space using dimensional regularization in D space-time dimensions. The Feynman rules are given in appendix A in ref. [23]. This diagram involves a triple gluon vertex, where two gluons attach at different points z_1 and z_2 along the time like Wilson line representing the incoming heavy quark (with momentum $p_\mu = mn_\mu$; $n^2 = 1$) and the third gluon attaches to the lightlike line going along the "$-$" direction at the point z_3.

We parametrize the points along the Wilson lines as in ref. [23], $z_1 = \tau_1 n$, $z_2 = \tau_2 n$ and $z_3 = \tau_3 y$, and denote the triple gluon vertex by z_4, obtaining:

$$W_{11} = \frac{-i}{2} g^4 C_A C_F \int_{-\infty}^{0} d\tau_1 \int_{\tau_1}^{0} d\tau_2 \int_{0}^{1} d\tau_3 \, \Big(y - (n \cdot y) n\Big) \cdot \left(\frac{d}{dz_1} - \frac{d}{dz_2}\right) \mathcal{J}(z_i) \tag{D.1}$$

with

$$\mathcal{J}(z_i) \equiv \int d^D z_4 \prod_{i=1}^{3} \mathbf{D}(z_i - z_4) \tag{D.2}$$

where the Feynman gauge propagator $\mathbf{D}(z)$ is:

$$\mathbf{D}(z) = \int \frac{d^D k}{(2\pi)^D} \, \mathrm{e}^{-ik\cdot z} \frac{i}{k^2 + i\varepsilon} = \frac{\Gamma(D/2-1)}{4\pi^{D/2}} \Big[- z^2 + i\varepsilon \Big]^{1-D/2} . \tag{D.3}$$

The calculation of $\mathcal{J}(z_i)$ using Feynman parametrization is straightforward and yields:

$$\mathcal{J}(z_i) = \frac{-i\Gamma(D-3)}{4^3 \pi^D} \int_0^1 dt \int_0^{1-t} ds \, [st(1-s-t)]^{D/2-2} \, (-\Delta)^{3-D} \tag{D.4}$$

with

$$\Delta \equiv t(1-t)z_1^2 + s(1-s)z_2^2 + (s+t)(1-s-t)z_3^2 - \\ -2stz_1 \cdot z_2 - 2s(1-s-t)z_2 \cdot z_3 - 2t(1-s-t)z_1 \cdot z_3 \\ = \tau_1^2 t(1-t) - 2st\tau_1\tau_2 + \tau_2^2 s(1-s) - 2n \cdot y(1-s-t)\tau_3\,[s\tau_2 + t\tau_1]\,. \tag{D.5}$$

Taking the derivative in eq. (D.1) and scaling $\tau_{1,2}$ by $-2n \cdot y$ one obtains:

$$W_{11} = \frac{1}{2} g^4 C_A C_F \frac{\Gamma(D-2)(2in \cdot y)^{8-2D}}{2^7 \pi^D} \int_0^1 ds \int_0^{1-s} dt (1-s-t)^{D/2-1} (st)^{D/2-2} (s-t) \times \\ \times \int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^1 d\tau_3 \tau_3 \Big[\tau_1^2 t(1-t) - 2st\tau_1\tau_2 + \tau_2^2 s(1-s) + \\ + (1-s-t)\tau_3 \left(s\tau_2 + t\tau_1\right) \Big]^{2-D} .$$

By first changing the integration variable τ_2 into σ, where $\tau_2 = \sigma\tau_1$, and then the integration variable τ_1 into ω, where

$$\tau_1 = \omega \frac{(1-s-t)\tau_3(\sigma s + t)}{\sigma^2 s(1-s) - 2st\sigma + t(1-t)} ,$$

one finds that the τ_3 integration is trivial, giving a factor of $1/(8-2D)$, while the ω-integration yields:

$$\int_0^\infty d\omega \omega^{3-D} (1+\omega)^{2-D} = -\frac{4^D \Gamma(D-5/2)\Gamma(3-D)}{128\sqrt{\pi}} .$$

The result can be expressed as:

$$W_{11} = \frac{1}{2} g^4 C_A C_F g^4 \frac{(2in \cdot y)^{8-2D}}{4^2 \pi^D} I_{11} , \tag{D.6}$$

where I_{11} matches the notation of ref [23]. We have: $I_{11} = K_{11} \times J_{11}$ with

$$J_{11} = \int_0^1 d\sigma \int_0^1 ds \int_0^{1-s} dt\, (1-s-t)^{5-\frac{3}{2}D} (st)^{\frac{D}{2}-2} (s-t) \times \\ \times (\sigma s + t)^{6-2D} \left(\sigma^2 s(1-s) - 2st\sigma + t(1-t)\right)^{D-4} , \tag{D.7}$$

and

$$K_{11} = -\frac{4^{D-5}\Gamma(D-2)\Gamma(D-5/2)\Gamma(3-D)}{(8-2D)\sqrt{\pi}} . \tag{D.8}$$

Proceeding with the evaluation of J_{11} we change the integration variable t into z, where $t = (1-s)z/(1+z)$, and then, after changing the order of integration, σ into x where $x = (1+z)(1-\sigma)$ and s into y where $y = s/z$, getting:

$$J_{11} = \int_0^\infty dz (1+z)^{D-5} \int_0^{1+z} dx \int_0^{1/z} dy y^{D/2-2} (y(1+2z) - 1) \times \\ \times \Big(1 + y(1-x)\Big)^{6-2D} \Big(1 + y(1-x)^2\Big)^{D-4} . \tag{D.9}$$

Since the dependence of the integrand on z is simple we perform this integration first. The price is having several terms as the z-integration extends between $\max\{0, x-1\}$ and $1/y$. Fortunately, most of the terms cancel out by symmetry and we obtain:

$$\begin{aligned} J_{11} = & \frac{-2}{D-3}\int_0^\infty dy \int_1^{(1+y)/y} dx y^{D/2-1} x^{D-3}\Big(1+y(1-x)\Big)^{6-2D}\Big(1+y(1-x)^2\Big)^{D-4} + \\ & +\frac{1}{D-4}\int_0^\infty dy \int_1^{(1+y)/y} dx (1+y) y^{D/2-2} x^{D-4}\Big(1+y(1-x)\Big)^{6-2D} \times \\ & \qquad \times \Big(1+y(1-x)^2\Big)^{D-4}. \end{aligned} \tag{D.10}$$

By changing the integration variable x into z where $z = 1 + (1-z)/y$ and then y into w where $y = (1-z)/w$, the two integrals can be computed exactly, yielding:

$$\begin{aligned} J_{11} = & \frac{-2}{D-3}\frac{\pi}{\sin(\pi D/2)}\frac{\Gamma(6-3D/2)}{\Gamma(3-D/2)\Gamma(6-D)} + \\ & +\frac{1}{D-4}\frac{1}{(2D-7)}\frac{\pi}{\sin(\pi D/2)} \times \\ & \times \left[{}_3F_2\left(\begin{matrix} 4-D, 7-2D, 2-D/2 \\ 8-2D, 6-3D/2 \end{matrix}, 1 \right) - {}_3F_2\left(\begin{matrix} 4-D, 7-2D, 1-D/2 \\ 8-2D, 7-3D/2 \end{matrix}, 1 \right) \right], \end{aligned} \tag{D.11}$$

where the first line is the result of the first integral in eq. (D.10) while the remaining terms, containing hypergeometric functions, correspond to the second.

Combining eqs. (D.11) and (D.8) we obtain the final result for I_{11}, which can be readily expanded in ϵ, where $D = 4 - 2\epsilon$. The result is:

$$\begin{aligned} I_{11} = & \frac{1}{192}\epsilon^{-4} + \left(-\frac{1}{96}+\frac{1}{96}\gamma_E\right)\epsilon^{-3} + \left(-\frac{1}{48}+\frac{13}{1152}\pi^2+\frac{1}{96}\gamma_E^2-\frac{1}{48}\gamma_E\right)\epsilon^{-2} + \\ & +\left(\frac{19}{288}\zeta_3+\frac{13}{576}\pi^2\gamma_E+\frac{1}{144}\gamma_E^3-\frac{1}{24}-\frac{1}{48}\gamma_E^2-\frac{1}{24}\gamma_E-\frac{13}{576}\pi^2\right)\epsilon^{-1}+\mathcal{O}(1). \end{aligned} \tag{D.12}$$

Finally, in terms of the renormalized coupling, eq. (D.6) takes the form:

$$\begin{aligned} W_{11} &= \frac{1}{2}C_A C_F\left(\frac{\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^2 (in\cdot y\mu e^{\frac{1}{2}\gamma_E})^{4\epsilon} I_{11} \\ &= \frac{1}{2}C_A C_F\left(\frac{\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^2 e^{4L\epsilon}e^{-2\gamma_E\epsilon} I_{11}, \end{aligned} \tag{D.13}$$

where in the first line we introduced the $\overline{\rm MS}$ factorization and renormalization scale according to eq. (3.39) and in the second we defined $L \equiv \ln ip\cdot y\tilde{\mu}/m = \ln N\tilde{\mu}/m$ with $\tilde{\mu} \equiv e^{\gamma_E}\mu$, as in eq. (3.23), knowing that the ϵ-expansion of $e^{-2\gamma_E\epsilon}I_{11}$ is free of γ_E terms. Performing this expansion, and subtracting the $\epsilon \longrightarrow 0$ singular terms, we finally get:

$$\begin{aligned} W_{11}^{\rm ren.} = & \frac{1}{2}C_A C_F\left(\frac{\alpha_s^{\overline{\rm MS}}(\mu)}{\pi}\right)^2 \left[\frac{1}{18}L^4 - \frac{1}{9}L^3 + \left(\frac{13}{144}\pi^2 - \frac{1}{6}\right)L^2 + \right. \\ & \left. + \left(\frac{19}{72}\zeta_3 - \frac{1}{6} - \frac{13}{144}\pi^2\right)L + \mathcal{O}(1)\right], \end{aligned} \tag{D.14}$$

which differs from eq. (3.6) in ref. [23] by the single-log term only.

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